INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 40, Northern Spring 2019 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. A positive integer divisible by 7 is shown on a computer screen. The cursor marks a gap between some pair of its consecutive digits. Prove that there is a digit that can be inserted into the marked gap any number of times so that the resulting number is always divisible by 7. (5 points)

2. There are 2019 crickets sitting on a straight line. Consider each cricket to be a point on the line. At each move one of the crickets jumps over one of the other crickets, landing at a point that is the same distance away from that cricket as before the jump. Jumping to the right only, the crickets are able to position themselves so that some pair of them are located exactly 1 mm from each other. Prove that the crickets are also able to position themselves so that two of them are exactly 1 mm apart, with the crickets jumping to the left only, and starting from the same initial position. (6 points)

3. Two equal non-intersecting wooden discs, one coloured grey and the other black, are in a fixed position of the plane. A wooden triangle with one grey edge and one black edge can be moved in the plane so that the discs remain outside the triangle while the coloured edges of the triangle are tangent to the discs of the same colour (the points of tangency not being the vertices). Prove that it is possible to draw a ray emanating from the vertex of the angle and inside the angle so that no matter how the angle is positioned, the ray passes through a fixed point in the plane. (7 points)

4. It is desired that all the squares of an $n \times n$ table ($n > 1$) be filled with distinct integers from 1 to n^2, such that there is one number per square, and each pair of consecutive integers are placed in squares that share a side, while any pair of integers having the same remainder on division by n are placed in distinct rows and distinct columns. For which n is this possible? (8 points)

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5. The orthogonal projection of a tetrahedron onto a plane containing one of its faces is a trapezium of area 1.

(a) Is it possible that the orthogonal projection of the tetrahedron onto a plane containing one of its other faces is a square of area 1? (4 points)

(b) The same question as (a) for a square of area 1/2019. (4 points)

6. Petya and Vasya play a game with a pile of cards. For each subset of five different variables from the set $\{x_1, \ldots, x_{10}\}$ there is a single card with their product written on the card. With Petya starting, Petya and Vasya alternate choosing one card from the pile of cards. After all cards have been drawn from the pile, Vasya assigns numerical values to the variables as he wants, except that he must ensure $0 \leq x_1 \leq \cdots \leq x_{10}$. Can Vasya make his assignations in such a way that ensures the sum of the products on his cards is greater than the sum of the products on Petya's cards? (8 points)

7. On the grid plane all possible broken lines with the following properties are constructed: each broken line starts at the point $(0, 0)$, it has all its vertices at points with integer coordinates and each line segment either goes up or to the right along the grid lines. For each broken line consider a corresponding *worm*, which is a shape consisting of grid squares that share at least one point with the broken line. Prove that the number of worms that can be divided into dominoes (2×1 or 1×2 rectangles) in exactly $n > 2$ different ways, is equal to the number of positive integers that are less than n and relatively prime to n. (Worms are different, if they consist of different sets of grid squares.) (12 points)